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Receptance Method in Active Vibration Control

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The pole/zero assignment problem is addressed using a method based on measured receptances. The approach, which is well known in passive structural modification, has not been used before in active vibration control. A number of significant advantages are claimed over the conventional state-space approach that uses the mass, damping, and stiffness matrices formed, for example, by finite elements. In fact, because the method is based solely upon measured vibration data, there is no need to evaluate or to know the M, C, and K matrices. It is demonstrated that all the poles may be assigned actively by the equivalent of a rank-1 modification to the dynamic stiffness matrix of the system. The assignment of zeros has a special significance in vibration suppression, because the vibration response at coordinate p vanishes completely when sinusoidal excitation is applied at coordinate q at the frequency of a zero of receptance H_{pq}. A pole of H_{pq} may be eliminated by assigning a zero at the same frequency.

Nomenclature

$\mathbf{A}$	=	system matrix in state-space form
$\mathbf{b}$	=	force selection vector
$\mathbf{C}$	=	damping matrix
$\mathbf{d}$	=	input vector in state-space form
$\mathbf{f}$, $\mathbf{g}$	=	vectors of state-feedback gains
$\mathbf{H}(s)$	=	receptance transfer function
$\mathbf{H}(i\omega)$	=	receptance frequency response function
$\mathbf{K}$	=	stiffness matrix
$\mathbf{M}$	=	mass matrix
$\mathbf{p}$	=	disturbance
s	=	complex Laplace frequency
t	=	time
u	=	control input
$\mathbf{z}$	=	state vector
μ_k	=	kth eigenvalue
ω	=	radian frequency

I. Introduction

THE problem of eigenvalue assignment has received considerable attention from the active-control and vibrations communities over several decades [1]. The placement of poles and zeros (natural frequencies and antiresonances) has many potential applications in structural dynamics, including the avoidance of damaging large-amplitude vibrations close to resonance and the design of adaptive structures capable of changing their behavior to respond in a desirable way to a varying demand. Passive modification can be traced back to the work of Duncan [2], who determined in 1941 that the dynamic behavior of a compound system formed from two or more subsystems with known receptances and interconnection properties. A typical inverse problem might be to assign a number of poles and zeros by adding structural elements such as point masses, springs, beams, or plates. In this way, the natural frequencies of a structure may be shifted to desired locations or antiresonances may be moved so that the vibration response vanishes at chosen coordinates and frequencies [3,4]. The advantage of the structural modification approach is that the system is guaranteed to remain stable. However, there are very considerable disadvantages:

1) The form of the modification that can be realized physically (symmetry, positive definiteness, reciprocity, and pattern of nonzero matrix terms) is restrictive.

2) Rotational receptances [3] are very difficult to measure and require high levels of specialist expertise.

3) The number of eigenvalues to be assigned must be matched by the rank of the modification.

Eigenvalue assignment in the active-control community dates from the 1960s when Wonham [5] showed that poles of a system could be assigned by state feedback if the system was controllable. Kautsky et al. [6] described numerical methods for determining robust (well-conditioned) solutions to the state-feedback pole assignment problem by defining a solution space of linearly independent eigenvectors, corresponding to the required eigenvalues. The solutions obtained were such that the sensitivity of assigned poles to perturbations in the system and gain matrices was minimized. Of particular interest in active vibration control is the quadratic eigenvalue problem (QEP) considered by Tisseur and Meerbergen [7], who described the various linearizations (transformations of the QEP into linear generalized eigenvalue problems with the same eigenvalues) and methods of computation, as well as including a catalogue of available software. The following papers deal with closed-loop control of dynamic systems described by the quadratic pencil with symmetric $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ matrices. Chu and Datta [8] showed that the displacement and velocity feedback gain matrices contained strictly real terms when self-conjugate poles were assigned. Datta et al. [9] developed a closed-form solution for the partial pole assignment problem, in which chosen eigenvalues were relocated and all the other eigenvalues were left unchanged. Henrion et al. [10] applied matrix inequality constraints to the controller by convex optimization for robust stability. Duan [11] developed two approaches based on singular value decomposition and right factorization, resulting in simple parametric expressions for the feedback gains and closed-loop eigenvectors. Qian and Xu [12] used a method that minimized the condition number of the eigenvector matrix for robust partial eigenvalue assignment. Similar techniques have been used in model updating [13]. Lin and Wang [14] used an output feedback technique and Carvalho et al. [15] used

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low-rank symmetric modifications to adjust a numerical model by partial pole placement.

In this paper, a new theory is presented for state feedback in active vibration suppression, based upon measured receptances, without any need to evaluate or know the $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ matrices. The approach has many virtues, as will now be explained. Although in principle, finite elements and conventional state-space analysis provide an ideal theoretical framework for active vibration control, finite element (FE) models used in design calculations are, in several important respects, incompatible with the practicalities of state-space vibration control.

1) Variational methods provide a sound theoretical basis for FE stiffness and mass matrices, but there exists no equivalent approach that can deal with the many different forms of damping (viscous, material, friction, impact, etc.) that are known to exist in real structures. FE models often neglect damping completely or else assume an ad hoc Rayleigh (or proportional) damping without physical justification. In conventional lightly damped structural vibration studies, the lack of a damping model (or use of an approximate one) is often acceptable, but in active control, the damping model is vitally important to complex eigenvalue analysis and to an understanding of relative stability.

2) FE models used in design (for stress analysis, impact, and dynamics) are usually very large, typically hundreds of thousands of degrees of freedom and quite often millions of degrees of freedom. In practice, model reduction methods are applied, but truncations and other approximations are generally introduced that can result in degraded controller performance.

3) The controller must be sufficiently insensitive to ill-defined FE parameters such as joints and boundary conditions. The parameters usually lose their physical meaning when model reduction is applied and uncertainty may then become "smeared" throughout the model, possibly resulting in lack of robustness.

The new theory is explained in detail and the method demonstrated by means of a series of simple example problems. Characteristic equations, linear in the unknown system gains, are formulated for the assignment of poles, zeros, or a combination of poles and zeros. In principle, a single actuator may be used to assign all the poles of the system. The problem of assigning a selected subset of poles (not the same as partial pole placement) is treated in another paper [16] that is also based on measured receptances, but using output feedback with collocated actuators and sensors instead of state feedback as developed in the present paper.

II. Theory and Examples

The principles of the active vibration control by the receptance method may be formulated as follows:

Consider the general $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ system with state feedback,

$$\begin{gathered}\mathbf{M}\,\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{b}u(t) + \mathbf{p}(t) \\ u(t) = -\mathbf{f}^T\dot{\mathbf{x}}(t) - \mathbf{g}^T\mathbf{x}(t)\end{gathered} \tag{1}$$

where $\mathbf{M}, \mathbf{C}, \mathbf{K} \in \Re^{n\times n}$; $\mathbf{M} = \mathbf{M}^T$, $\mathbf{C} = \mathbf{C}^T$, $\mathbf{K} = \mathbf{K}^T$; $\mathbf{v}^T\mathbf{M}\mathbf{v} > 0$, $\mathbf{v}^T\mathbf{C}\mathbf{v} \geq 0$, and $\mathbf{v}^T\mathbf{K}\mathbf{v} \geq 0$ for arbitrary $\mathbf{v} \neq \mathbf{0}$, $\mathbf{v} \in \Re^{n\times 1}$; and $\mathbf{b}, \mathbf{p}, \mathbf{f}, \mathbf{g} \in \Re^{n\times 1}$. It should be noted that, in practice, each nonzero term in $\mathbf{b}$ implies the use of an actuator and each nonzero term in $\mathbf{g}$ or $\mathbf{f}$ implies the use of a sensor.

Then, by combining Eqs. (1) and expressing the result in the frequency domain,

$$[\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K} + \mathbf{b}(\mathbf{g} + s\mathbf{f})^T]\mathbf{x}(s) = \mathbf{p}(s) \tag{2}$$

it is seen that the closed-loop dynamic stiffness is changed by the rank-1 matrix $\mathbf{b}(\mathbf{g} + s\mathbf{f})^T$, which is a consequence of state feedback using a single input $u(t)$.

The Sherman–Morrison formula [17] gives the inverse of a matrix with a rank-1 modification in terms of the inverse of the original matrix. Thus, the closed-loop receptance matrix is found to be

$$\hat{\mathbf{H}}(s) = \mathbf{H}(s) - \frac{\mathbf{H}(s)\mathbf{b}(\mathbf{g} + s\mathbf{f})^T\mathbf{H}(s)}{1 + (\mathbf{g} + s\mathbf{f})^T\mathbf{H}(s)\mathbf{b}} \tag{3}$$

where $\mathbf{H}(s) = [\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K}]^{-1}$, in practice, may be determined from the matrix of *measured* receptances $\mathbf{H}(i\omega)$ at the sensor/actuator coordinates. Further discussion of Eq. (3) may be found in [18], in which it is shown that certain terms in $\hat{\mathbf{H}}(s)$ become indeterminate (zero divided by zero) as s approaches a pole of $\mathbf{H}(s)$. In practice, $\hat{\mathbf{H}}(s)$ becomes sensitive to measurement noise.

A. Pole-Assignment Problem

The characteristic polynomial of the closed-loop system is $1 + (\mathbf{g} + s\mathbf{f})^T\mathbf{H}(s)\mathbf{b}$, and the problem of assigning the poles of the system to predetermined values $\{\mu_1, \mu_2, \ldots, \mu_{2n}\}$ may be expressed as follows:

Given $\mathbf{H}(s)$, $\mathbf{b}$, and a complex set $\{\mu_1, \mu_2, \ldots, \mu_{2n}\}$, closed under conjugation, find $\mathbf{g}$ and $\mathbf{f}$, such that $(\mathbf{g} + \mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\mathbf{b} = -1$ for $k = 1, \ldots, 2n$.

To solve the problem, we denote

$$\mathbf{r}_k = \mathbf{H}(\mu_k)\mathbf{b} \tag{4}$$

Then we need to solve

$$\mathbf{r}_k^T(\mathbf{g} + \mu_k\mathbf{f}) = -1, \qquad k = 1, \ldots, 2n$$

or

$$\mathbf{r}_k^T\mathbf{g} + \mu_k\mathbf{r}_k^T\mathbf{f} = -1, \qquad k = 1, \ldots, 2n \tag{6}$$

The set of $2n$ equations with $2n$ unknowns may be written in matrix form:

$$\begin{bmatrix} \mathbf{r}_1^T & \mu_1\mathbf{r}_1^T \\ \mathbf{r}_2^T & \mu_2\mathbf{r}_2^T \\ \vdots & \vdots \\ \mathbf{r}_{2n}^T & \mu_{2n}\mathbf{r}_{2n}^T \end{bmatrix} \begin{pmatrix} \mathbf{g} \\ \mathbf{f} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} \tag{7}$$

which allows the determination of $\mathbf{g}$ and $\mathbf{f}$ by inversion of the matrix

$$\mathbf{G} = \begin{bmatrix} \mathbf{r}_1^T & \mu_1\mathbf{r}_1^T \\ \mathbf{r}_2^T & \mu_2\mathbf{r}_2^T \\ \vdots & \vdots \\ \mathbf{r}_{2n}^T & \mu_{2n}\mathbf{r}_{2n}^T \end{bmatrix} \tag{8}$$

Theorem 1: $\mathbf{G}$ is invertible if the system is controllable and μ_k and $k = 1, \ldots, 2n$ and are distinct.

Proof: The pole placement problem defined here may be transformed to the following standard form:

Given $\mathbf{A}$, $\mathbf{d}$, and the set $\{\mu_1, \mu_2, \ldots, \mu_{2n}\}$, find vector $\boldsymbol{\beta}$, such that the eigenvalues of

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{d}u(t) \tag{9}$$

are the prescribed set $\{\mu_1, \mu_2, \ldots, \mu_{2n}\}$, where

$$u(t) = \mathbf{d}\boldsymbol{\beta}^T \tag{10}$$

and

$$\begin{gathered}\mathbf{z}(t) = \begin{pmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{pmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \\ \mathbf{d} = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{b} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\beta} = \begin{pmatrix} \mathbf{g} \\ \mathbf{f} \end{pmatrix}\end{gathered} \tag{11}$$

The well-known classical solution is given by Ackermann [19]:

$$\boldsymbol{\beta} = -\mathbf{P}^T\boldsymbol{\Psi}^{-1}\mathbf{e}_{2n} \tag{12}$$

where

$$\mathbf{P} = \mathrm{A}^{2n} + \alpha_{2n-1}\mathrm{A}^{2n-1} + \alpha_{2n-2}\mathrm{A}^{2n-2} + \cdots + \alpha_0 \mathbf{I} \tag{13}$$

the coefficients α_i are the coefficients of the desired characteristic polynomial

$$\mu^{2n} + \alpha_{2n-1}\mu^{2n-1} + \alpha_{2n-2}\mu^{2n-2} + \cdots + \alpha_0 = (\mu - \mu_{2n})(\mu - \mu_{2n-1})\cdots(\mu - \mu_1) \tag{14}$$

and $\boldsymbol{\Psi}$ is the controllability matrix

$$\boldsymbol{\Psi} = [\,\mathbf{d} \quad \mathbf{Ad} \quad \mathbf{A}^2\mathbf{d} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{d}\,]^T \tag{15}$$

Hence, whenever $\boldsymbol{\Psi}$ is controllable (i.e, the inverse $\boldsymbol{\Psi}^{-1}$ exists), there is a solution to the pole assignment problem. This solution can be achieved by Eq. (7) without the explicit knowledge of **M**, **C**, and **K**. If, however, μ_k is a required eigenvalue of multiplicity p, then it is easy to see that **G** has p repeated equations and, hence, **G** is not invertible in this case, but the system of equations in Eq. (7) is still consistent.

Theorem 2: If **G** is invertible and the set $\{\mu_1, \mu_2, \ldots, \mu_{2n}\}$ is closed under conjugation, then **g** and **f** are real.

Proof: We write Eq. (7) in the form

$$\mathbf{G}\,\boldsymbol{\beta} = \boldsymbol{\gamma} \tag{16}$$

with the obvious definition of **G**, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$. Assume that the vector $\boldsymbol{\beta} = (\,\mathbf{g}^T \quad \mathbf{f}^T\,)^T$ has imaginary components so that $\boldsymbol{\beta} = \boldsymbol{\beta}_R + i\boldsymbol{\beta}_I$ and, similarly, $\mathbf{G} = \mathbf{G}_R + i\mathbf{G}_I$, where $\boldsymbol{\beta}_R$, $\boldsymbol{\beta}_I$, $\mathbf{G}_R$, and $\mathbf{G}_I$ are real. Equation (16) thus takes the form

$$(\mathbf{G}_R + i\mathbf{G}_I)(\boldsymbol{\beta}_R + i\boldsymbol{\beta}_I) = \boldsymbol{\gamma} \tag{17}$$

Let us now substitute the complex conjugate set $\{\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_{2n}\}$ in Eq. (7). We note that because $\{\mu_1, \mu_2, \ldots, \mu_{2n}\}$ is self-conjugate, such a substitution has the effect of merely rearranging the equations in Eq. (7). Hence, under such substitution, the solution for **f** and **g** (or, equivalently, $\boldsymbol{\beta}$) remains invariant. We therefore write the system of equations corresponding to this case as

$$(\mathbf{G}_R - i\mathbf{G}_I)(\boldsymbol{\beta}_R + i\boldsymbol{\beta}_I) = \boldsymbol{\gamma} \tag{18}$$

From Eq. (17), we obtain

$$\mathbf{G}_R\boldsymbol{\beta}_R - \mathbf{G}_I\boldsymbol{\beta}_I = \boldsymbol{\gamma} \tag{19}$$

and

$$\mathbf{G}_R\boldsymbol{\beta}_I + \mathbf{G}_I\boldsymbol{\beta}_R = \mathbf{0} \tag{20}$$

From Eq. (18), we obtain

$$\mathbf{G}_R\boldsymbol{\beta}_R + \mathbf{G}_I\boldsymbol{\beta}_I = \boldsymbol{\gamma} \tag{21}$$

and

$$\mathbf{G}_R\boldsymbol{\beta}_I - \mathbf{G}_I\boldsymbol{\beta}_R = \mathbf{0} \tag{22}$$

Subtracting Eq. (19) from (21),

$$2\mathbf{G}_I\boldsymbol{\beta}_I = \mathbf{0} \tag{23}$$

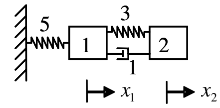

Fig. 1 Mass-spring-damper system.

and adding Eq. (20) to Eq. (22),

$$2\mathbf{G}_R\boldsymbol{\beta}_I = \mathbf{0} \tag{24}$$

Equations (23) and (24) then give

$$(\mathbf{G}_R + i\mathbf{G}_I)\boldsymbol{\beta}_I = \mathbf{G}\boldsymbol{\beta}_I = \mathbf{0} \tag{25}$$

According to the previous Theorem, **G** is invertible whenever the system is controllable and μ_k, $k = 1, 2, \ldots, 2n$ are distinct. It thus follows that, in this case, $\boldsymbol{\beta}_I = \mathbf{0}$.

Example 1: With $\mathbf{b} = (\,1 \quad 2\,)^T$, we wish to assign the poles of the system shown in Fig. 1 to $\mu_1 = -1 + 10i$, $\mu_2 = -1 - 10i$, $\mu_3 = -2$, and $\mu_4 = -3$.

Here,

$$\mathbf{H}\,(s) = (s^2\mathbf{M} + s\mathbf{C} + \mathbf{K})^{-1}$$

where

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} 8 & -3 \\ -3 & 3 \end{bmatrix}$$

From Eq. (4), we obtain

$$\mathbf{r}_1 = [(-1+10i)^2\mathbf{M} + (-1+10i)\mathbf{C} + \mathbf{K}]^{-1}\mathbf{b} = \begin{pmatrix} -0.0102 + 0.0021i \\ -0.0097 + 0.0020i \end{pmatrix}$$

$$\mathbf{r}_2 = [(-1-10i)^2\mathbf{M} + (-1-10i)\mathbf{C} + \mathbf{K}]^{-1}\mathbf{b} = \begin{pmatrix} -0.0102 - 0.0021i \\ -0.0097 - 0.0020i \end{pmatrix}$$

$$\mathbf{r}_3 = [(-2)^2\mathbf{M} + (-2)\mathbf{C} + \mathbf{K}]^{-1}\mathbf{b} = \begin{pmatrix} 0.1236 \\ 0.2360 \end{pmatrix}$$

$$\mathbf{r}_4 = [(-3)^2\mathbf{M} + (-3)\mathbf{C} + \mathbf{K}]^{-1}\mathbf{b} = \begin{pmatrix} 0.0714 \\ 0.1111 \end{pmatrix}$$

so that by using Eq. (8),

$$\mathbf{G} = \begin{bmatrix} -0.0102 + 0.0021i & -0.0097 + 0.0020i & -0.0110 - 0.1043i & -0.0101 - 0.0989i \\ -0.0102 - 0.0021i & -0.0097 - 0.0020i & -0.0110 + 0.1043i & -0.0101 + 0.0989i \\ 0.1236 & 0.2360 & -0.2472 & -0.4719 \\ 0.0714 & 0.1111 & -0.2143 & -0.3333 \end{bmatrix}$$

The set of linear Eqs. (7) then gives

$$\mathbf{g} = \begin{pmatrix} 68.8750 \\ 30.3750 \end{pmatrix} \qquad \mathbf{f} = \begin{pmatrix} -62.6750 \\ 68.1750 \end{pmatrix}$$

To validate the results, we obtain the eigenvalues of the closed-loop system $\mathbf{A} - \lambda\mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -(\mathbf{K} + \mathbf{bg}^T) & -(\mathbf{C} + \mathbf{bf}^T) \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}$$

and find, as required, that they are $\mu_1 = -1 + 10i$, $\mu_2 = -1 - 10i$, $\mu_3 = -2$, and $\mu_4 = -3$.

B. Zero Assignment Problem

The characteristic polynomial of the zeros of receptance $\hat{H}_{ij}$ is given from the ijth numerator term of Eq. (3).

Given $\mathbf{H}(s)$, $\mathbf{b}$, i, j, and a complex set $\{\mu_1 \quad \mu_2 \quad \cdots \quad \mu_r\}$, find $\mathbf{g}$ and $\mathbf{f}$, such that

$$\mathbf{e}_i^T\left(\mathbf{H}(\mu_k) - \frac{\mathbf{H}(\mu_k)\mathbf{b}(\mathbf{g}+\mu_k\mathbf{f})^T\mathbf{H}(\mu_k)}{1+(\mathbf{g}+\mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\mathbf{b}}\right)\mathbf{e}_j = 0$$

for $k = 1, \ldots, r$, where $\mathbf{e}_k$ is the unit vector formed from the kth column of the identity matrix and $r \le 2(n-1)$.

The ijth term in the transfer function matrix of the closed-loop system is given from Eq. (3) as

$$\hat{H}_{ij}(s) = \frac{\mathbf{e}_i^T[(1+(\mathbf{g}+s\mathbf{f})^T\mathbf{H}(s)\mathbf{b})\mathbf{H}(s) - \mathbf{H}(s)\mathbf{b}(\mathbf{g}+s\mathbf{f})^T\mathbf{H}(s)]\mathbf{e}_j}{1+(\mathbf{g}+s\mathbf{f})^T\mathbf{H}(s)\mathbf{b}} \tag{26}$$

We wish to find $\mathbf{f}$ and $\mathbf{g}$, such that

$$\mathbf{e}_i^T\{[1+(\mathbf{g}+\mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\mathbf{b}]\mathbf{H}(\mu_k) - \mathbf{H}(\mu_k)\mathbf{b}(\mathbf{g} + \mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\}\mathbf{e}_j = 0 \tag{27}$$

for $k = 1, 2, \ldots, r$. Equation (27) may be written in the form

$$[(\mathbf{g}+\mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\mathbf{b}]\mathbf{e}_i^T\mathbf{H}(\mu_k)\mathbf{e}_j - \mathbf{e}_i^T[\mathbf{H}(\mu_k)\mathbf{b}(\mathbf{g} + \mu_k\mathbf{f})^T\mathbf{H}(\mu_k)]\mathbf{e}_j = -\mathbf{e}_i^T\mathbf{H}(\mu_k)\mathbf{e}_j \tag{28}$$

or

$$H_{ij}(\mu_k)(\mathbf{g}+\mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\mathbf{b} - \mathbf{e}_i^T\mathbf{H}(\mu_k)\mathbf{b}(\mathbf{g}+\mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\mathbf{e}_j = -H_{ij}(\mu_k) \tag{29}$$

with the obvious definition of $H_{ij}(s)$. Noting that $\mathbf{e}_i^T\mathbf{H}(\mu_k)\mathbf{b}$ is a scalar, we define

$$\mathbf{t}_k = H_{ij}(\mu_k)\mathbf{H}(\mu_k)\mathbf{b} - [\mathbf{e}_i^T\mathbf{H}(\mu_k)\mathbf{b}]\mathbf{H}(\mu_k)\mathbf{e}_j \tag{30}$$

and write the linear system (29) for $k = 1, \ldots, r$ in matrix form,

$$\begin{bmatrix} \mathbf{t}_1^T & \mu_1\mathbf{t}_1^T \\ \mathbf{t}_2^T & \mu_2\mathbf{t}_2^T \\ \vdots & \vdots \\ \mathbf{t}_r^T & \mu_r\mathbf{t}_r^T \end{bmatrix}\begin{pmatrix}\mathbf{g}\\ \mathbf{f}\end{pmatrix} = \begin{pmatrix} -H_{ij}(\mu_1) \\ -H_{ij}(\mu_2) \\ \vdots \\ -H_{ij}(\mu_r)\end{pmatrix} \tag{31}$$

which allows determination of $\mathbf{g}$ and $\mathbf{f}$ as functions, depending on $2n - r$ arbitrary constants. Of course, only real $\mathbf{g}$ and $\mathbf{f}$ are realizable control parameters.

Theorem 3: If $i = j$ and the set $\{\mu_1,\ \mu_2, \ldots, \mu_r\}$ is closed under conjugation, then $\mathbf{g}$ and $\mathbf{f}$ can be chosen as real vectors.

Proof: As with the Proof of Theorem 2, we write Eq. (31) in the form

$$\mathbf{G}\boldsymbol{\beta} = \boldsymbol{\gamma} \tag{32}$$

so that $\boldsymbol{\beta} = \boldsymbol{\beta}_R + i\boldsymbol{\beta}_I$, $\mathbf{G} = \mathbf{G}_R + i\mathbf{G}_I$, and in this case, $\boldsymbol{\gamma} = \boldsymbol{\gamma}_R + i\boldsymbol{\gamma}_I$; where $\boldsymbol{\gamma}_R$ and $\boldsymbol{\gamma}_I$, are real. Equation (32) then takes the form

$$(\mathbf{G}_R + i\mathbf{G}_I)(\boldsymbol{\beta}_R + i\boldsymbol{\beta}_I) = \boldsymbol{\gamma}_R + i\boldsymbol{\gamma}_I \tag{33}$$

Now we substitute the complex conjugate set $\{\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_{2n}\}$ in Eq. (31) and note that because $\{\mu_1, \mu_2, \ldots, \mu_{2n}\}$ is self-conjugate, the substitution has the effect of rearranging Eqs. (31). Hence, under such a substitution, the solution for $\mathbf{f}$ and $\mathbf{g}$ (and, equivalently, $\boldsymbol{\beta}$) remains invariant. We therefore write the system of equations corresponding to this case as

$$(\mathbf{G}_R - i\mathbf{G}_I)(\boldsymbol{\beta}_R + i\boldsymbol{\beta}_I) = \boldsymbol{\gamma}_R - i\boldsymbol{\gamma}_I \tag{34}$$

From Eq. (33), we obtain

$$\mathbf{G}_R\boldsymbol{\beta}_R - \mathbf{G}_I\boldsymbol{\beta}_I = \boldsymbol{\gamma}_R \tag{35}$$

and

$$\mathbf{G}_R\boldsymbol{\beta}_I + \mathbf{G}_I\boldsymbol{\beta}_R = \boldsymbol{\gamma}_I \tag{36}$$

From Eq. (34), we obtain

$$\mathbf{G}_R\boldsymbol{\beta}_R + \mathbf{G}_I\boldsymbol{\beta}_I = \boldsymbol{\gamma}_R \tag{37}$$

and

$$\mathbf{G}_R\boldsymbol{\beta}_I - \mathbf{G}_I\boldsymbol{\beta}_R = -\boldsymbol{\gamma}_I \tag{38}$$

Subtracting Eq. (35) from Eq. (37),

$$2\mathbf{G}_I\boldsymbol{\beta}_I = \mathbf{0} \tag{39}$$

and adding Eq. (36) to Eq. (38),

$$2\mathbf{G}_R\boldsymbol{\beta}_I = \mathbf{0} \tag{40}$$

Equations (39) and (40) then give

$$(\mathbf{G}_R + i\mathbf{G}_I)\boldsymbol{\beta}_I = \mathbf{G}\boldsymbol{\beta}_I = \mathbf{0} \tag{41}$$

and we may, therefore, choose $\boldsymbol{\beta}_I = \mathbf{0}$.

It was shown in [20] that when $i = j$, the zero assignment problem is equivalent to the pole assignment problem of the system, with the constraint that the ith degree of freedom is restricted to have no motion. Hence, in this case, the solution to the zero assignment problem exists.

Example 2: With $\mathbf{b} = (1 \quad 2)^T$, we wish to assign the zeros of $H_{22}(s)$ of the system shown in Fig. 1 to $\mu_1 = -1$ and $\mu_2 = -2$.

From Eq. (30), we obtain

$$\mathbf{t}_1 = \begin{pmatrix} 0.0357 \\ 0 \end{pmatrix} \qquad \mathbf{t}_2 = \begin{pmatrix} 0.0112 \\ 0 \end{pmatrix}$$

so that by Eq. (31),

$$\mathbf{G} = \begin{bmatrix} 0.0357 & 0 & -0.0357 & 0 \\ 0.0112 & 0 & -0.0225 & 0 \end{bmatrix}$$

The minimal norm solution of Eq. (31) then gives

$$\mathbf{g} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} \qquad \mathbf{f} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

To validate the results, we obtain the transfer functions of the closed-loop system,

$$\mathbf{H}(-1) = [(-1)^2\mathbf{M} + (-1)(\mathbf{C} + \mathbf{b}\mathbf{f}^T) + \mathbf{K} + \mathbf{b}\mathbf{g}^T]^{-1} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \left[\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}(2 \quad 0)\right] + \begin{bmatrix} 8 & -3 \\ -3 & 3 \end{bmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}(-6 \quad 0)\right\}^{-1} = \begin{bmatrix} 0 & -2 \\ -18 & 4 \end{bmatrix}^{-1} = \frac{1}{18}\begin{bmatrix} -2 & -1 \\ -9 & 0 \end{bmatrix}$$

and

$$\mathbf{H}(-2) = [(-2)^2\mathbf{M} + (-2)(\mathbf{C} + \mathbf{b}\mathbf{f}^T) + \mathbf{K} + \mathbf{b}\mathbf{g}^T]^{-1}$$
$$= \left\{4\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - 2\left[\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}(2 \quad 0)\right]\right.$$
$$\left. + \begin{bmatrix} 8 & -3 \\ -3 & 3 \end{bmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}(-6 \quad 0)\right\}^{-1} = \begin{bmatrix} 0 & -1 \\ -21 & 9 \end{bmatrix}^{-1}$$
$$= \frac{1}{21}\begin{bmatrix} -9 & -1 \\ -21 & 0 \end{bmatrix}$$

and find, as requested, that $H_{22}(-1) = H_{22}(-2) = 0$.

C. Assignment of Poles and Zeros

Given $\mathbf{H}(s)$, $\mathbf{b}$, i, j, and two complex sets $\{\mu_1 \quad \mu_2 \quad \cdots \quad \mu_r\}$ and $\{\xi_1 \quad \xi_2 \quad \cdots \quad \xi_p\}$, find $\mathbf{g}$ and $\mathbf{f}$, such that

$$\mathbf{e}_i^T\left(\mathbf{H}(\mu_k) - \frac{\mathbf{H}(\mu_k)\mathbf{b}(\mathbf{g} + \mu_k\mathbf{f})^T\mathbf{H}(\mu_k)}{1 + (\mathbf{g} + \mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\mathbf{b}}\right)\mathbf{e}_j = 0, \qquad k = 1, \ldots, r$$
$$(\mathbf{g} + \xi_k\mathbf{f})^T\mathbf{H}(\xi_k)\mathbf{b} = -1, \qquad k = 1, \ldots, p$$

where $r + p \leq 2n$.

Solve

$$\begin{bmatrix} \mathbf{t}_1^T & \mu_1\mathbf{t}_1^T \\ \mathbf{t}_2^T & \mu_2\mathbf{t}_2^T \\ \vdots & \vdots \\ \mathbf{t}_r^T & \mu_r\mathbf{t}_r^T \\ \mathbf{r}_1^T & \xi_1\mathbf{r}_1^T \\ \mathbf{r}_2^T & \xi_2\mathbf{r}_2^T \\ \vdots & \vdots \\ \mathbf{r}_p^T & \xi_p\mathbf{r}_p^T \end{bmatrix}\begin{pmatrix} \mathbf{g} \\ \mathbf{f} \end{pmatrix} = \begin{pmatrix} -H_{ij}(\mu_1) \\ -H_{ij}(\mu_2) \\ \vdots \\ -H_{ij}(\mu_r) \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} \tag{42}$$

where $\mathbf{t}_k$ and $k = 1, \ldots, r$ are given by Eq. (30) and

$$\mathbf{r}_k = \mathbf{H}(\xi_k)\mathbf{b}, \qquad k = 1, \ldots, p \tag{43}$$

Example 3: With $\mathbf{b} = (1 \quad 2)^T$, we wish to assign the poles and the system in Fig. 1 and the zeros of $H_{22}(s)$ to $\mu_1 = \xi_1 = 6i$ and $\mu_2 = \xi_2 = -6i$, thereby eliminating the pole at 6 rad/s, with a zero at the same frequency.

From Eqs. (42) and (43), it is found that

$$\mathbf{G} = \begin{bmatrix} 0.00047 + 0.000151i & 0 & -0.00091 + 0.002828i & 0 \\ 0.00047 - 0.000151i & 0 & -0.00091 - 0.002828i & 0 \\ -0.03242 - 0.001059i & -0.02771 + 0.000456i & 0.006353 - 0.19453i & -0.002735 - 0.16625i \\ -0.03242 + 0.001059i & -0.02771 - 0.000456i & 0.006353 + 0.19453i & -0.002735 + 0.16625i \end{bmatrix}$$

resulting in the solution

$$\mathbf{g} = \begin{pmatrix} 28 \\ 3 \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and assigning the undamped poles and zeros together, as requested, at $\pm 6i$.

Two further poles of the closed-loop system appear at $-0.75 \pm 1.984i$

The second natural frequency, corresponding to the undamped poles at $\pm 6i$, is almost invisible from Fig. 2a, but can be clearly observed in Fig. 2b, because the zeros of H_{11} and H_{22} are not the same. The small blip appearing in Fig. 2a is due to digital machine calculation with a finite register of significant figures. Because the pole-zero cancellation occurs exactly on the imaginary axis, the result is that a vibration node is assigned at the second mass to an undamped mode.

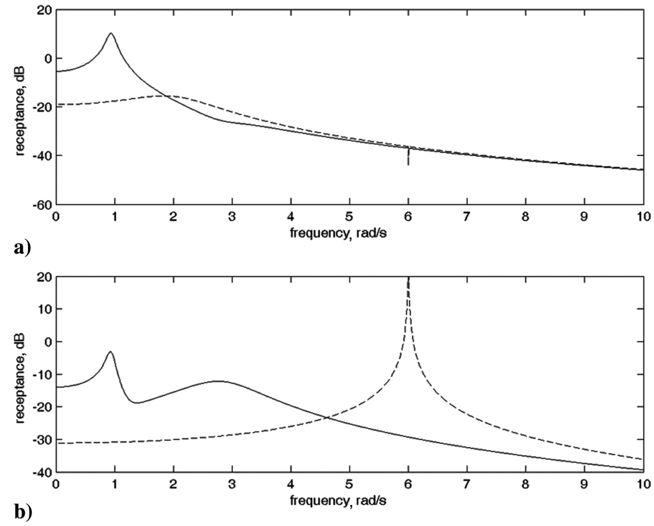

Fig. 2 Original (solid line) and assigned (dashed line) receptances: a) H_{22} and b) H_{11}.

III. Conclusions

A new theory is developed for the assignment of poles and zeros in active vibration control using the receptance method, and its application is demonstrated by means of a series of simple example problems. In the work presented, the receptance transfer function is determined by inverting the dynamic stiffness matrix, but, in practice, it would be obtained from the measured receptance frequency response function $H(i\omega)$, and so there would be no need for the evaluation of the system matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ that are typically obtained from finite elements. The method leads to linear characteristic equations in the unknown gains, a consequence of the state feedback used, resulting in a rank-1 change to the dynamic stiffness matrix.

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